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Entropy Generation and Integral Inequalities

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As a typical irreversible process, the heat conduction in rectangles results in entropy generation. As time tends to infinity, this entropy generation evolves into a finite value when the heat conduction comes from the initial temperature distribution, but into infinity whenever a positively-averaged heat source is involved. An application of the second law of thermodynamics to this process leads to nine integral inequalities which are important for studying heat-conduction equations and for uncovering some basic features of the total multiplicity and the Boltzmann entropy. The work correlates the second law of thermodynamics in thermodynamics and integral inequalities in mathematics, and inspires the future work in offering some fundamental insights into our future.

Keywords: Entropy generation; Integral inequalities; Second law of thermodynamics; Heat conduction

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1. Introduction

Energy is defined as the ability or the potential of system to cause changes.[1, 2] It can exist in numerous forms such as thermal, mechanical, kinetic, potential, electric, magnetic, chemical, and nuclear; their sum constitutes the total energy of a system.[1, 2] During every practical process, energy is conserved following the first law of thermodynamics, but degraded according to the second law of thermodynamics.[1, 2] The conservation in energy quantity is represented by the heat-conduction equation for heat-conduction processes in which the classical Fourier's law of heat conduction is normally used as the constitutive relation of heat flux density.[3-5] The degradation comes from the entropy generation occurring in any practical process like heat conduction because of irreversibilities.[1, 2, 6]

The solution of the heat-conduction equation with its initial and boundary conditions yields the temperature field, the temperature at any location and any time instant in heat conduction.[3, 4] The Fourier's law of heat conduction is then applied to find heat-transfer rate and the way to enhance/reduce it.[3, 4] With the known temperature field, the entropy generation in heat conduction becomes available as well by using the thermodynamic relations from the first and second laws of thermodynamics.[1, 2, 6] With the available entropy generation, we can then develop integral inequalities and examine features of heat-conduction equations by applying the second law of thermodynamics in the form that *the entropy generation S_{gen} of a system during heat conduction always increases, or, in the limiting case of a reversible process, remains constant*, i.e., $dS_{gen}/dt \geq 0$ *with t being the time* and $S_{gen}(t_2) \geq S_{gen}(t_1)$ for all $t_2 \geq t_1$.[1, 2, 6] The entropy at state 1 can be defined as, $S_1 = CQ_{0R}^{10}$, where Q_{0R}^{10} is the heat exchanged between the system and the environment undergoing a totally reversible process from State 1 to the reference state 0, and C can be any positive constants.[7] Q_{0R}^{10} is process-independent and represents the part of system energy that cannot be used to do useful work.[7] Constant C was introduced for recovering the classical definition of entropy. However, it is more convenient and desirable for illustrating the physical meaning of entropy and performing entropy analysis with C being 1 (no unit) such that entropy being with an energy unit. This definition is identical to the classical one by choosing the constant C as the inverse of the environment temperature (so that the classical definition of entropy with a unit of J/K).[7] Any entropy generation will thus downgrade the energy quality. It is therefore important to study the entropy generation in general, and its boundedness in particular.

We examine dS_{gen}/dt and $\lim_{t\to\infty} S_{gen}$ and derive mathematical inequalities with heat conduction in rectangles. This kind of analysis is recently initiated with heat conduction in cylinders,[8] is limited in the literature[8] and differs from conventional second-law analysis that has been made mainly for performance evaluation, weak-component identification or performance optimization.[9-30]

2. Temperature field, entropy generation and integral inequalities

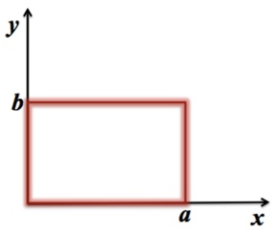

Fig. 1 Heat conduction in adiabatic rectangles and Cartesian coordinate system.

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Consider heat conduction in a rectangle D of width a and height b subject to specified temperature gradient atthe rectangle boundary. Material properties are assumed to be constant. Note that the effect of nonhomogeneous boundary condition is representable by source and initial terms.[3, 6] We thus limit our attention to the following initial-boundary value problem under homogeneous boundary conditions in Cartesian coordinates, shown in Fig. 1, without loss of the generality:[3, 4, 31]

$$\begin{cases} T_t = a_0^2\left(T_{xx} + T_{yy}\right) + f(x,y,t),\ \mathrm{D} \times (0, +\infty) \\ T_x|_{x=0,\ a} = T_y|_{y=0,\ b} = 0 \\ T|_{t=0} = \varphi(x,y) \end{cases} \tag{1}$$

$$\mathrm{D}: 0 < x < a, 0 < y < b$$

where t and T are time and temperature, respectively. a_0^2 is the thermal diffusivity. $\varphi\ (x, y)$ is the initial temperature distribution over the rectangle. $f\ (x, y, t)$ is the rate of volumetric heat generation inside the rectangle per unit specific capacity of the material. The heat generation may be due to chemical, electrical,gammy-ray, nuclear, or other sources that may be a function of time and/or position. Fig. 2 examples the heat conduction driven by the initial temperature distribution, the internal source and the both, respectively, the readers are refer to[31] for more practical examples.

2.1 Heat conduction initiated by the initial temperature distribution

For the heat conduction driven by the initial temperature distribution, $f\ (x, y, t) = 0$, and Eq. (1) reduces into

$$\begin{cases} T_t = a_0^2\left(T_{xx} + T_{yy}\right),\ \mathrm{D} \times (0, +\infty) \\ T_x|_{x=0,\ a} = T_y|_{y=0,\ b} = 0 \\ T|_{t=0} = \varphi(x,y) \end{cases} \tag{2}$$

Applying generalized Fourier extension to the solution of (2) yields,[3]

$$T(x,y,t) = \sum_{m,n=0}^{\infty} T(t)\, a_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{3}$$

Because

$$T_t = \sum_{m,n=0}^{\infty} T_t(t)\, a_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{4}$$

$$T_{xx} = \sum_{m,n=0}^{\infty} -\left(\frac{m\pi}{a}\right)^2 T(t)\, a_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{5}$$

$$T_{yy} = \sum_{m,n=0}^{\infty} -\left(\frac{n\pi}{b}\right)^2 T(t)\, a_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{6}$$

Eq. (2) becomes

$$\sum_{m,n=0}^{\infty} T_t(t) a_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} + \sum_{m,n=0}^{\infty} \left(\frac{m\pi a_0}{a}\right)^2 T(t) a_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} + \sum_{m,n=0}^{\infty} \left(\frac{n\pi a_0}{b}\right)^2 T(t)\, a_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} = 0 \tag{7}$$

Let $\omega_{mn}^2 = \left(\frac{m\pi a_0}{a}\right)^2 + \left(\frac{n\pi a_0}{b}\right)^2$, thus

$$\sum_{m,n=0}^{\infty} a_{mn} \left[T_t(t) + \omega_{mn}^2 T(t)\right] \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} = 0 \tag{8}$$

Therefore, one has, because $a_{mn} \neq 0$,

$$T_t(t) + \omega_{mn}^2 T(t) = 0 \tag{9}$$

The solution of Eq. (9) reads,[3]

$$T_{mn}(t) = c_{mn} e^{-\omega_{mn}^2 t},\ c_{mn} \text{ is constant} \tag{10}$$

Define constant, $A_{mn} = a_{mn} c_{mn}$then substituting Eq. (10) into Eq. (3) yields the solution of Eq. (2)

$$T(x,y,t) = \sum_{m,n=0}^{\infty} A_{mn} e^{-\omega_{mn}^2 t} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{11}$$

By using the initial condition and the orthogonality of function group $\left\{\cos\frac{m\pi x}{a}\right\}, x \in (0,a)$ and $\left\{\cos\frac{n\pi y}{b}\right\}, y \in (0,b)$, one arrives at

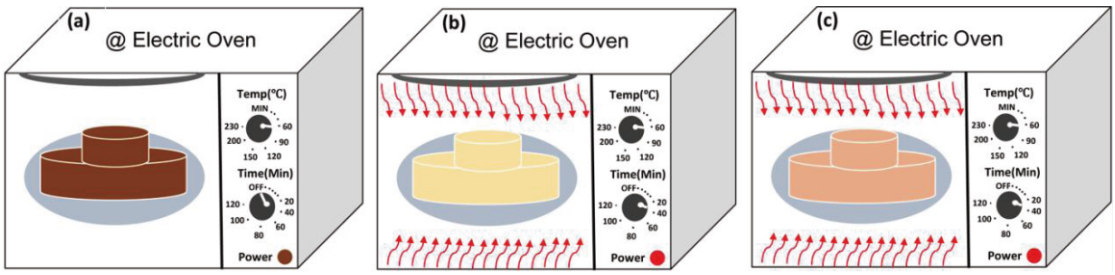

Fig. 2 Heat conduction in oven: (a) heat conduction driven by the initial temperature distribution after switching off the power, (b) heat conduction driven by the internal source when the power is first switched on, (c) heat conduction driven by the initial temperature distribution and the internal source after re-switching on the power.

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$$\varphi(x,y) = \sum_{m,n=0}^{\infty} A_{mn} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{12}$$

$$A_{mn} = \frac{1}{M_{mn}}\int_0^b\int_0^a \varphi(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}dxdy \tag{13}$$

where

$$M_{mn} = \begin{cases} ab, & m = n = 0 \\ \dfrac{ab}{2}, & m = 0, n = 1 \text{ or } m = 0, n = 1 \\ \dfrac{ab}{4}, & m,n = 1,2,3,\ldots \end{cases} \tag{14}$$

Therefore, the temperature distribution $T_{\varphi}(x,y,t)$ due to initial temperature distribution $\varphi(x,y)$ can be expressed as

$$\begin{cases} T_{\varphi}(x,y,t) = \sum_{m,n=0}^{\infty} A_{mn} e^{-\omega_{mn}^2 t}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ A_{mn} = \frac{1}{M_{mn}}\iint_{\mathrm{D}} \varphi(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}d\mathrm{A}\ , m,n = 0,1,2,\ldots \end{cases} \tag{15}$$

where $A = ab$ is the area of domain D.

The total entropy in the rectangle is, by using the first ds equation (the one of two fundamental equations for calculating the entropy change in engineering thermodynamics) from the first and second law of thermodynamics,[1, 2, 6]

$$S_{\varphi}(t) = \rho C_V \int_0^b\int_0^a \ln\left[T_{\varphi}(x,y,t)\right]dx\,dy$$

$$= \rho C_V \iint_{\mathrm{D}} \ln\left(\sum_{m,n=0}^{\infty} A_{mn} e^{-\omega_{mn}^2 t}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right)d\mathrm{A} \tag{16}$$

Note that $S_{\varphi}(t)$ is also the total entropy generation for the case of adiabatic boundary conditions with vanished thermal entropy flux.[1, 6]

By applying Eq. (16) and noting that $\omega_{mn}^2 > 0$ for $m + n \neq 0$, we obtain

$$\lim_{t\to+\infty} S_{\varphi}(t)$$

$$= \rho C_V \int_0^b\int_0^a \lim_{t\to+\infty} \ln\left(\sum_{m,n=0}^{\infty} A_{mn} e^{-\omega_{mn}^2 t}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right)dx\,dy$$

$$= \rho C_V \int_0^b\int_0^a \lim_{t\to+\infty} \ln\left(A_{00} + \sum_{m,n=0;\, m+n\neq 0}^{\infty} A_{mn} e^{-\omega_{mn}^2 t}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right)dx\,dy$$

$$= \rho C_V \int_0^b\int_0^a \ln A_{00}\,dx\,dy = \rho C_V \iint_D \ln A_{00}\,dA \tag{17}$$

By using Eqs. (13) and (14),

$$A_{00} = \frac{1}{A}\iint_{\mathrm{D}} \varphi(x,y)d\mathrm{A} \tag{18}$$

which is the average of the initial temperature distribution $\varphi(x, y)$ over the rectangle. Therefore,

$$\lim_{t\to+\infty} S_{\varphi}(t)$$

$$= \rho C_V \iint_D \ln\left[\frac{1}{A}\iint_{\mathrm{D}} \varphi(x,y)d\mathrm{A}\right]dA$$

$$= \rho C_V A \ln\left[\frac{1}{A}\iint_{\mathrm{D}} \varphi(x,y)d\mathrm{A}\right] \tag{19}$$

The limit of $S_{\varphi}(t)$ is thus bounded as time trends to infinity and equals to system total entropy with the temperature being the average of the initial temperature distribution over the whole rectangle. Note that,

$$\frac{dS_{\varphi}(t)}{dt} = \rho C_v \int_0^b\int_0^a \frac{1}{T_{\varphi}(x,y,t)}\frac{\partial T_{\varphi}(x,y,t)}{\partial t}dx\,dy$$

$$\frac{\partial T_{\varphi}(x,y,t)}{\partial t} = \sum_{m,n=0}^{\infty} A_{mn}(-\omega_{mn}^2) e^{-\omega_{mn}^2 t}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{20}$$

Applying the principle of entropy increase $dS_{\varphi}(t)/dt \geq 0$ leads to,

$$\iint_{\mathrm{D}} \frac{\sum_{m,n=0}^{\infty} A_{mn}(-\omega_{mn}^2) e^{-\omega_{mn}^2 t}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}}{\sum_{m,n=0}^{\infty} A_{mn} e^{-\omega_{mn}^2 t}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}}\,d\mathrm{A} \geq 0 \tag{21}$$

As $dS_{\varphi}(t)/dt \geq 0$, one also has

$$S_{\varphi}(t_1) \leq S_{\varphi}(t_2) \quad \forall 0 < t_1 \leq t_2 < +\infty \tag{22}$$

or

$$\int_0^b\int_0^a \ln\left[T_{\varphi}(x,y,t_1)\right]dx\,dy \leq \int_0^b\int_0^a \ln\left[T_{\varphi}(x,y,t_2)\right]dxdy \tag{23}$$

This yields the integral inequality

$$\iint_{\mathrm{D}} \ln\left(\sum_{m,n=0}^{\infty} A_{mn} e^{-\omega_{mn}^2 t_1}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right)d\mathrm{A}$$

$$\leq \iint_{\mathrm{D}} \ln\left(\sum_{m,n=0}^{\infty} A_{mn} e^{-\omega_{mn}^2 t_2}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right)d\mathrm{A} \tag{24}$$

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Note that

$$S_\varphi(0)=\rho C_V\int_0^b\int_0^a \ln\left[T_\varphi(x,y,t)\right]dxdy=\rho C_V\iint_D \ln\varphi(x,y)\,dA \quad (25)$$

and

$$S_\varphi(+\infty)=\rho C_V A\ln\left[\frac{1}{A}\iint_D \varphi(x,y)dA\right] \quad (26)$$

Applying $S_\varphi(0)\le S_\varphi(+\infty)$ leads to

$$\rho C_V\iint_D \ln\varphi(x,y)\,dA \le \rho C_V A\ln\left[\frac{1}{A}\iint_D \varphi(x,y)dA\right] \quad (27)$$

that is

$$\frac{1}{A}\iint_D \ln\varphi(x,y)\,dA\le\ln\left[\frac{1}{A}\iint_D \varphi(x,y)dA\right] \quad (28)$$

which is the two-dimensional extension over the rectangle of well-known Arithmetic-mean---geometric-mean inequality[32, 33] and can be proven mathematically as following as well.

By the third law of thermodynamics, $\varphi(x,y)=T_\varphi(x,y,0)>0$ with T_φ being the temperature during the heat conduction driven exclusively by the initial temperature distribution $\varphi(x,y)$. Divide D into N area elements $\Delta\sigma_i, i=1,2,...,N,$ of equal area. For $\forall(\xi_i,\eta_i)\in\Delta\sigma_i$, one has N positive numbers:

$$\varphi(\xi_i,\eta_i),\ i=1,2,...,N \quad (29)$$

Therefore,

$$\sqrt[N]{\prod_{i=1}^N \varphi(\xi_i,\eta_i)}\le\frac{\sum_{i=1}^N\varphi(\xi_i,\eta_i)}{N} \quad (30)$$

and

$$\ln\sqrt[N]{\prod_{i=1}^N \varphi(\xi_i,\eta_i)}\le\ln\frac{\sum_{i=1}^N\varphi(\xi_i,\eta_i)}{N} \quad (31)$$

or

$$\frac{1}{N}\sum_{i=1}^N\ln\varphi(\xi_i,\eta_i)\le\ln\frac{\sum_{i=1}^N\varphi(\xi_i,\eta_i)}{N} \quad (32)$$

which can be written as

$$\frac{1}{A}\left[\sum_{i=1}^N\ln\varphi(\xi_i,\eta_i)\right]\frac{A}{N}\le\ln\left[\frac{1}{A}\sum_{i=1}^N\varphi(\xi_i,\eta_i)\frac{A}{N}\right] \quad (33)$$

Note that $A/N=\Delta x_i\Delta y_i$, one has

$$\frac{1}{A}\left[\sum_{i=1}^N\ln\varphi(\xi_i,\eta_i)\,\Delta x_i\Delta y_i\right]\le\ln\left[\frac{1}{A}\sum_{i=1}^N\varphi(\xi_i,\eta_i)\,\Delta x_i\Delta y_i\right] \quad (34)$$

$$\iint_D f(x,y)\,dA=\lim_{N\to\infty}\sum_{i=1}^N f(\xi_i,\eta_i)\,\Delta x_i\Delta y_i \quad (35)$$

and

$$\frac{1}{A}\lim_{N\to\infty}\left[\sum_{i=1}^N\ln\varphi(\xi_i,\eta_i)\Delta x_i\Delta y_i\right]\le\ln\left\{\frac{1}{A}\lim_{N\to\infty}\left[\sum_{i=1}^N\varphi(\xi_i,\eta_i)\Delta x_i\Delta y_i\right]\right\} \quad (36)$$

Therefore, one arrives at Eq. (26),

$$\frac{1}{A}\iint_D\ln\varphi(x,y)dA\le\ln\left[\frac{1}{A}\iint_D\varphi(x,y)dA\right] \quad (37)$$

While Eq. (28) is developed from the 2nd law of thermodynamics with $\varphi(x,y)$ being the initial temperature distribution, it is actually valid for any positive function $\varphi(x,y)$. Consider the total multiplicity Ω of a macrostate, the number of microstates that compose the macrostate, in the Boltzmann entropy $S = k\ln\Omega$ with k being the Boltzmann's constant.[34-36] Eq. (28) leads to,

$$\frac{1}{A}\iint_D\ln\Omega dA\le\ln\left[\frac{1}{A}\iint_D\Omega dA\right] \quad (38)$$

$$\frac{1}{A}\iint_D k\ln\Omega dA\le k\ln\left[\frac{1}{A}\iint_D\Omega dA\right] \quad (39)$$

Therefore, the average of entropy over any domain can never be larger than the entropy calculated with the average of multiplicity over the domain.

2.2 Heat conduction driven by the internal source

For the heat conduction driven exclusively by the internal source, $\varphi(x,y) = 0$ and Eq. (1) reduces

$$\begin{cases}T_t=a_0^2\left(T_{xx}+T_{yy}\right)+f(x,y,t),\ D\times(0,+\infty)\\ T_x|_{x=0,\ a}=T_y|_{y=0,\ b}=0\\ T|_{t=0}=0\end{cases} \quad (40)$$

$$D{:}0<x<a,0<y<b$$

Note that,[3]

$$f(x,y,t)=\int_0^t f(x,y,\tau)\,\delta(\tau-t)d\tau$$

where δ stands for the δ function. The solution of Eq. (40) is thus, by the superposition principle,[3]

$$T_f(x,y,t)=\int_0^t U(x,y,t,\tau)d\tau \quad (41)$$

in which $U(x,y,t,\tau)$is the solution of

$$\begin{cases}U_t=a_0^2\left(U_{xx}+U_{yy}\right)+f(x,y,\tau)\delta(\tau-t)\\ U_x|_{x=0,\ a}=U_y|_{y=0,\ b}=0\\ U|_{t=0}=0\end{cases} \quad (42)$$

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or

$$\begin{cases} U_t = a_0^2\left(U_{xx} + U_{yy}\right) \\ U_x|_{x=0,\ a} = U_y|_{y=0,\ b} = 0 \\ U|_{t=\tau} = f(x,y,\tau) \end{cases} \tag{43}$$

By replacing t and $\varphi(x,y)$ in Eq. (15) with $t - \tau$ and $f(x,y,\tau)$, respectively, the solution of Eq. (42) reads,

$$U(x,y,t,\tau) = \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2 (t-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{44}$$

Thus,

$$\begin{cases} T_f(x,y,t) = \int_0^t \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2 (t-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} d\tau \\ B_{mn} = \frac{1}{M_{mn}} \int_0^b \int_0^a f(\xi,\eta,\tau)\cos\frac{m\pi\xi}{a}\cos\frac{n\pi\eta}{b} d\xi d\eta, m,n = 0,1,2,3,\ldots \end{cases} \tag{45}$$

where T_f stands for the temperature for heat conduction driven exclusively by the internal heat source and can also be rewritten as,[3]

$$T_f(x,y,t) = \int_0^t \int_0^b \int_0^a G(x,\xi,y,\eta,t-\tau) f(\xi,\eta,\tau) d\xi d\eta d\tau \tag{46}$$

where

$$G(x,\xi,y,\eta,t-\tau) = \sum_{m,n=0}^{\infty} e^{-\omega_{mn}^2 (t-\tau)} \cos\frac{m\pi x}{a}\cos\frac{m\pi\xi}{a}\cos\frac{n\pi y}{b}\cos\frac{n\pi\eta}{b} \tag{47}$$

is the Green function[3].

The total entropy in the rectangle at time instant t is, which is also the entropy generation up to time instant t,[1,2,6]

$$S_f(t) = \rho C_V \int_0^b \int_0^a \ln\left[T_f(x,y,t)\right] dx\, dy$$

$$= \rho C_V \iint_D \ln\left[\int_0^t \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2 (t-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} d\tau\right] dA \tag{48}$$

By applying Eq.(48)

$$\lim_{t\to+\infty} S_f(t) = \rho C_v \int_0^b \int_0^a \ln\left[\lim_{t\to+\infty} T_f(x,y,t)\right] dx\, dy$$

$$= \rho C_V \iint_D \ln\left\{\lim_{t\to+\infty}\left[\int_0^t \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2 (t-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} d\tau\right]\right\} dA \tag{49}$$

which can be unbounded for some internal heat sources [see the proof in the part regarding $S_f(+\infty)$, Eq. (70)].

Also,

$$\frac{dS_f(t)}{dt} = \rho C_v \int_0^b \int_0^a \frac{1}{T_f(x,y,t)} \frac{\partial T_f(x,y,t)}{\partial t} dx\, dy \geq 0 \tag{50}$$

Let $F(x,y,t,\tau) = \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2 (t-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$. By applying the rule of taking derivatives of integral with respect to its parameters, one has

$$\frac{\partial T_f(x,y,t)}{\partial t} = \int_0^t \frac{\partial F(x,y,t,\tau)}{\partial t} d\tau + F(x,y,t,\tau)|_{\tau=t} \tag{51}$$

where

$$\frac{\partial F(x,y,t,\tau)}{\partial t} = \sum_{m,n=0}^{\infty} B_{mn}(-\omega_{mn}^2) e^{-\omega_{mn}^2 (t-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} = \sum D_{mn}(-\omega_{mn}^2) e^{-\omega_{mn}^2 (t-\tau)} \tag{52}$$

$$\sum D_{mn} = \sum_{m,n=0}^{\infty} B_{mn} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{53}$$

$$F(x,y,t,\tau)|_{\tau=t} = f(x,y,t) \tag{54}$$

One thus has

$$\frac{\partial T_f(x,y,t)}{\partial t} = \int_0^t \sum D_{mn}(-\omega_{mn}^2) e^{-\omega_{mn}^2 (t-\tau)} d\tau + f(x,y,t) \tag{55}$$

By the second law of thermodynamics,

$$\frac{dS_f(t)}{dt} \geq 0$$

One has thus, by noting that $\rho C_V > 0$,

$$\iint_D \frac{\int_0^t \sum D_{mn}(-\omega_{mn}^2) e^{-\omega_{mn}^2 (t-\tau)} d\tau + f(x,y,t)}{\int_0^t \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2 (t-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} d\tau} dA \geq 0 \tag{56}$$

As $dS_f(t)/dt \geq 0$, one also has

$$S_f(t_1) \leq S_f(t_2) \quad \forall 0 < t_1 \leq t_2 < +\infty \tag{57}$$

Note that

$$S_f(t_1) = \rho C_V \iint_D \ln\left[\int_0^{t_1} \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2 (t_1-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} d\tau\right] dA \tag{58}$$

$$S_f(t_2) = \rho C_V \iint_D \ln\left[\int_0^{t_1} \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2 (t_1-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} d\tau\right] dA \tag{59}$$

The following inequality is thus obtained:

$$\iint_D \ln\left[\int_0^{t_2} \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2 (t_2-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} d\tau\right] dA$$

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$$\leq \iint_{\mathrm{D}} \ln\left[\int_0^{t_2} \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2(t_2-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\, d\tau\right] d\mathrm{A} \quad (60)$$

Also,

$$S_f(0) = \lim_{t\to+0} S_f(0) = \rho C_V \int_0^b \int_0^a \lim_{t\to+0} \ln\left[T_f(x,y,t)\right] dxdy$$

$$= \rho C_V \iint_{\mathrm{D}} \ln\left[\lim_{t\to+0} T_f(x,y,t)\right] d\mathrm{A} = \rho C_V \iint_{\mathrm{D}} \ln[0]\, d\mathrm{A} = -\infty \quad (61)$$

$$S_f(+\infty) = \rho C_V \iint_{\mathrm{D}} \ln\left[\lim_{t\to+\infty} T_f(x,y,t)\right] d\mathrm{A}$$

$$= \rho C_V \iint_{\mathrm{D}} \ln\left[\lim_{t\to+\infty}\left[\int_0^t \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2(t-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\, d\tau\right]\right] d\mathrm{A} \quad (62)$$

As $S_f(0) \leq S_f(\infty)$, one has,

$$\rho C_V \iint_{\mathrm{D}} \ln\left[\lim_{t\to+\infty}\left[\int_0^t \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2(t-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\, d\tau\right]\right] d\mathrm{A} \geq -\infty \quad (63)$$

Let's now examine $S_f(\infty)$in details. By Eq.(41),

$$T_f(x,y,t) = \int_0^t U(x,y,t,\tau) d\tau,$$

where $U(x,y,t,\tau)$ is the solution of (40) and reads [Eq. (41)],

$$U(x,y,t,\tau) = \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2(t-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

As

$$\lim_{t\to+\infty} U(x,y,t,\tau) = B_{00} = \frac{1}{A}\iint_D f(x,y,\tau) dA \quad (64)$$

Therefore, for sufficiently large t_0,

$$T_f(x,y,t_0) = \int_0^{t_0} U(x,y,t,\tau)\, d\tau = \int_0^{t_0} \frac{1}{A}\iint_D f(x,y,\tau)\, dA\, d\tau \quad (65)$$

and

$$S_f(t_0) = \rho C_v \iint_D \ln\left[\int_0^{t_0} \frac{1}{A}\iint_D f(x,y,\tau) dA\, d\tau\right] dA$$

$$= \rho C_V A \ln\left[\int_0^{t_0} \frac{1}{A}\iint_D f(x,y,\tau) dA\, d\tau\right] \quad (66)$$

Consider now the source with a positive average over the rectangle so that for any time instant τ, there exists a positive value ε such that

$$\frac{1}{A}\iint_D f(x,y,\tau) dA > \varepsilon > 0, \quad (67)$$

or

$$\frac{1}{\mathrm{A}}\int_0^{t_0} d\tau \iint_D f(x,y,\tau) dA > \varepsilon t_0 \quad (68)$$

Thus,

$$S_f(t_0) > \rho C_v A \ln(\varepsilon t_0) \quad (69)$$

As $\varepsilon > 0$, εt_0 can be sufficiently large for sufficiently large t_0. Note also that ln$x$ always increases with x. Therefore, with Eq. (69),

$$\forall\ M > 0,\ \exists\ t_0,\ \text{when } t_0 < t,\ M < S_f(t)$$

That is

$$\lim_{t\to+\infty} S_f(t) = S_f(+\infty) = +\infty \quad (70)$$

2.3 Heat conduction driven by the initial temperature distribution and the internal source

The temperature field subjected to the effect of both initial temperature distribution and the internal source is, by the superimposition principle,

$$T_{\varphi f}(x,y,t) = T_{\varphi}(x,y,t) + T_f(x,y,t) \quad (71)$$

With Eqs. (15) and (45), we have

$$T_{\varphi f}(x,y,t) = \sum_{m,n=0}^{\infty} A_{mn} e^{-\omega_{mn}^2 t} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} + \int_0^t \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2(t-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} d\tau$$

$$= \sum C_{mn}\, e^{-\omega_{mn}^2 t} + \int_0^t \sum D_{mn}\, e^{-\omega_{mn}^2(t-\tau)} d\tau \quad (72)$$

where

$$\sum C_{mn} = \sum_{m,n=0}^{\infty} A_{mn} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b};$$

$$\sum D_{mn} = \sum_{m,n=0}^{\infty} B_{mn} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \quad (73)$$

The total entropy in the rectangle at time instant t is, which is also the entropy generation up to time instant t,[1,2,6]

$$S_{\varphi f}(t) = \rho C_V \int_0^b \int_0^a \ln\left[T_{\varphi f}(x,y,t)\right] dx\, dy$$

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$$= \rho C_V \iint_D \ln\left[\sum C_{mn}\, e^{-\omega_{mn}^2 t} + \int_0^t \sum D_{mn}\, e^{-\omega_{mn}^2 (t-\tau)} d\tau\right] dA \quad (74)$$

Also,

$$\lim_{t\to+\infty} S_{\varphi f}(t)$$

$$= \rho C_V \int_0^b \int_0^a \ln\left[A_{00} + \lim_{t\to+\infty} T_f(x,y,t)\right] dxdy$$

$$= \rho C_V \iint_D \ln\left[\frac{1}{A}\iint_D \varphi(x,y) dA + \lim_{t\to\infty}\int_0^t \sum D_{mn}\, e^{-\omega_{mn}^2 (t-\tau)} d\tau\right] dA \quad (75)$$

Note that

$$\frac{dS_{\varphi f}(t)}{dt} = \rho C_V \int_0^b \int_0^a \frac{\partial}{\partial t}\left[\ln T_{\varphi f}(x,y,t)\right] dx\, dy$$

$$= \rho C_V \iint_D \frac{1}{T_\varphi(x,y,t) + T_f(x,y,t)}\left[\frac{\partial T_\varphi(x,y,t)}{\partial t} + \frac{T_f(x,y,t)}{\partial t}\right] dA \quad (76)$$

By Eqs. (20) and (55), one has

$$\frac{\partial T_\varphi(x,y,t)}{\partial t} + \frac{\partial T_f(x,y,t)}{\partial t}$$

$$= \sum C_{mn}\,(-\omega_{mn}^2) e^{-\omega_{mn}^2 t} + \int_0^t \sum D_{mn}\,(-\omega_{mn}^2) e^{-\omega_{mn}^2 (t-\tau)} d\tau + f(x,y,t) \quad (77)$$

By the second law of thermodynamics that requires $dS_{\varphi f}(t)/dt \geq 0$, one has

$$\iint_D \frac{\sum C_{mn}\,(-\omega_{mn}^2) e^{-\omega_{mn}^2 t} + \int_0^t \sum D_{mn}\,(-\omega_{mn}^2) e^{-\omega_{mn}^2 (t-\tau)} d\tau + f(x,y,t)}{\sum C_{mn}\, e^{-\omega_{mn}^2 t} + \int_0^t \sum D_{mn}\, e^{-\omega_{mn}^2 (t-\tau)} d\tau} dA \geq 0 \quad (78)$$

Note that

$$S_{\varphi f}(t_1) = \rho C_V \int_0^b \int_0^a \ln\left[T_{\varphi f}(x,y,t_1)\right] dxdy = \rho C_V \iint_D \ln\left[T_{\varphi f}(x,y,t_1)\right] dA \quad (79)$$

$$S_{\varphi f}(t_2) = \rho C_V \int_0^b \int_0^a \ln\left[T_{\varphi f}(x,y,t_2)\right] dxdy = \rho C_V \iint_D \ln\left[T_{\varphi f}(x,y,t_2)\right] dA \quad (80)$$

By applying

$$S_{\varphi f}(t_1) \leq S_{\varphi f}(t_2), \quad \forall\, 0 < t_1 \leq t_2 < +\infty$$

one obtains

$$\iint_D \ln\left[\sum C_{mn}\, e^{-\omega_{mn}^2 t_1} + \int_0^t \sum D_{mn}\, e^{-\omega_{mn}^2 (t_1-\tau)} d\tau\right] dA \leq \iint_D \ln\left[\left[\sum C_{mn}\, e^{-\omega_{mn}^2 t_2} + \int_0^t \sum D_{mn}\, e^{-\omega_{mn}^2 (t_2-\tau)} d\tau\right]\right] dA \quad (81)$$

When $t_1 = 0$, $t_2 = +\infty$,

$$S(0) = \rho C_V \int_0^b \int_0^a \ln\left[T_{\varphi f}(x,y,0)\right] dx\, dy$$

$$= \rho C_V \iint_D \ln\left[\varphi(x,y)\right] dA \quad (82)$$

$$S_{\varphi f}(+\infty) = \rho C_V \int_0^b \int_0^a \ln\left[\lim_{t\to+\infty} T_{\varphi f}(x,y,t)\right] dx\, d_y$$

$$= \rho C_V \iint_D \ln\left\{\frac{1}{A}\iint_D \varphi(x,y) dA + \lim_{t\to\infty}\int_0^t \sum D_{mn}\, e^{-\omega_{mn}^2 (t-\tau)} d\tau\right\} dA \quad (83)$$

thus one has

$$\iint_D \ln\left[\varphi(x,y)\right] dA$$

$$\leq \iint_D \ln\left\{\frac{1}{A}\iint_D \varphi(x,y) dA + \lim_{t\to\infty}\int_0^t \sum D_{mn}\, e^{-\omega_{mn}^2 (t-\tau)} d\tau\right\} dA \quad (84)$$

Consider now the source with a positive average over the rectangle so that for any time instant τ, there exists a positive value ε such that

$$\frac{1}{A}\iint f(x,y,t)\, dA > \varepsilon > 0 \quad (85)$$

By using the results in 2.1 and 2.2 and for sufficiently large *t*,

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$$S_{\varphi f}(t) = \rho C_V \int_0^b \int_0^a \ln\left[T_\varphi(x,y,t) + T_f(x,y,t)\right] dx\, dy$$

$$= \rho C_V \iint_D \ln\left\{\frac{1}{A}\iint_D \varphi(x,y) dA + \left[\lim_{t\to\infty}\left[\int_0^t \frac{1}{A}\iint f(x,y,t) dA d\tau\right]\right]\right\} dA$$

$$\geq \rho C_V \iint_D \ln\left[\frac{1}{A}\iint_D \varphi(x,y) dA + \varepsilon t\right] dA \quad (86)$$

As $\varepsilon > 0$, εt can be sufficiently large for sufficiently large t. Note also that ln$x$ always increases with x. Eq. (86) thus leads to

$$\forall\ M > 0\ ,\ \exists\ t_0\ ,\ \text{when}\ \ t_0 < t\ ,\ \ M < S_{\varphi f}(t)$$

that is

$$\lim_{t\to+\infty} S_{\varphi f}(t) = S_f(+\infty) = +\infty \quad (87)$$

3. Discussion

With an adiabatic boundary, as time tends to infinity, the temperature would become uniform for the heat conduction driven only by the initial temperature distribution, but not when there is any positively-averaged internal source. As time tends to infinity, the entropy generation from the heat conduction in adiabatic rectangles has a bounded limit [Eq. (17)] when the heat conduction is initiated by the initial temperature distribution. This is not valid anymore [Eqs. (70) and (87)] when the heat conduction includes an internal source with a positive average over the rectangle. This result is also valid in other systems.[37] As the entropy generation degrades the energy quality,[7] the unbounded entropy generation will cause serious problems. We must thus constrain, as far as possible, the use and of technologies that depend on positive average heat generation, such as electric-resistance heating, nuclear fission and exothermic chemical reactions.

Table 1 lists the nine integral inequalities from applying the second law of thermodynamics to the heat conduction in rectangles. Despite of the similar form, they differ from their 1D counterparts in literature[6] due to different temperature fields. One of these inequalities [Eq. (28)] has also been proven mathematically. For their fundamental nature (from the three fundamental laws: the second law of thermodynamics, the first law of thermodynamics and the Fourier law of heat conduction via the heat-conduction equation), they are important both for studying heat-conduction equations and for assessing accuracy of solutions from analytical, numerical and experimental approaches. Application of Eq. (28) to the total multiplicity Ω of a macrostate, the number of microstates that compose the macrostate in the Boltzmann entropy, has also led to Eqs. (38) and (39) showing that the entropy average over any domain can never be larger than the entropy calculated with the average multiplicity over the domain. Table 2 lists and $\frac{1}{A}\iint_D \ln\varphi(x,y)\,dA$ with a = 1 m, b = 2 m for four typical $\ln\left[\frac{1}{A}\iint_D \varphi(x,y)dA\right]$ $\varphi(x,y)$, and further demonstrates the correctness of Eq. (28). Equations (28), (38) and (39) are classical and have been acquired by the other approach in the literature. The present work develops a new approach,

Table 1 Integral inequalities developed in present work.

Heat conduction	Second law of thermodynamics	Inequalities
By initial temperature distribution	$\frac{dS_\varphi(t)}{dt} \geq 0$	$\iint_D \frac{\sum_{m,n=0}^{\infty} A_{mn}(-\omega_{mn}^2) e^{-\omega_{mn}^2 t} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}}{\sum_{m,n=0}^{\infty} A_{mn} e^{-\omega_{mn}^2 t} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}} dA \geq 0$
	$S_\varphi(t_1) \leq S_\varphi(t_2)$	$\iint_D \ln\left(\sum_{m,n=0}^{\infty} A_{mn} e^{-\omega_{mn}^2 t_1} \cosh\frac{m\pi x}{a}\cosh\frac{n\pi y}{b}\right) dA \leq \iint_D \ln\left(\sum_{m,n=0}^{\infty} A_{mn} e^{-\omega_{mn}^2 t_2} \cosh\frac{m\pi x}{a}\cosh\frac{n\pi y}{b}\right) dA$
	$S_\varphi(0) \leq S_\varphi(+\infty)$	$\frac{1}{A}\iint_D \ln\varphi(x,y) dA \leq \ln\left[\frac{1}{A}\iint_D \varphi(x,y) dA\right]$
By internal source	$\frac{dS_f(t)}{dt} \geq 0$	$\iint_D \frac{\int_0^t \sum D_{mn}\left(-\omega_{mn}^2\right) e^{-\omega_{mn}^2 (t-\tau)} d\tau + f(x,y,t)}{\int_0^t \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2 (t-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} d\tau} dA \geq 0$
	$S_f(t_1) \leq S_f(t_2)$	$\iint_D \ln\left[\int_0^{t_1} \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2 (t_1-\tau)} \cosh\frac{m\pi x}{a}\cosh\frac{n\pi y}{b} d\tau\right] dA \leq \iint_D \ln\left[\int_0^{t_2} \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2 (t_2-\tau)} \cosh\frac{m\pi x}{a}\cosh\frac{n\pi y}{b} d\tau\right] dA$

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Table 1

	$S_f(0) \le S_f(\infty)$	$\iint_D \ln\left[\lim_{t\to+\infty}\left[\int_0^t \sum_{m,n=0}^{\infty} B_{mn} e^{-\omega_{mn}^2(t-\tau)} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} d\tau\right]\right] dA \ge -\infty$
By both initial temperature distribution and internal source	$\frac{dS_{\varphi f}(t)}{dt} \ge 0$	$\iint_D \frac{\sum C_{mn}\left(-\omega_{mn}^2\right)e^{-\omega_{mn}^2 t} + \int_0^t \sum D_{mn}\left(-\omega_{mn}^2\right)e^{-\omega_{mn}^2(t-\tau)}d\tau + f(x,y,t)}{\sum C_{mn}\, e^{-\omega_{mn}^2 t} + \int_0^t \sum D_{mn}\, e^{-\omega_{mn}^2(t-\tau)}d\tau} dA \ge 0$
	$S_{\varphi f}(t_1) \le S_{\varphi f}(t_2)$	$\iint_D \ln\left[\sum C_{mn}\, e^{-\omega_{mn}^2 t_1} + \int_0^{t_1} \sum D_{mn}\, e^{-\omega_{mn}^2(t_1-\tau)}d\tau\right] dA \le \iint_D \ln\left[\left[\sum C_{mn}\, e^{-\omega_{mn}^2 t_2} + \int_0^{t_2} \sum D_{mn}\, e^{-\omega_{mn}^2(t_2-\tau)}d\tau\right]\right] dA$
	$S_{\varphi f}(0) \le S_{\varphi f}(\infty)$	$\iint_D \ln[\varphi(x,y)]\, dA \le \iint_D \ln\left\{\frac{1}{A}\iint_D \varphi(x,y)dA + \lim_{t\to\infty}\int_0^t \sum D_{mn}\, e^{-\omega_{mn}^2(t-\tau)}d\tau\right\} dA$

Table 2 Comparison between $\frac{1}{A}\iint_D \ln\varphi(x,y)\,dA$ and $\ln\left[\frac{1}{A}\iint_D \varphi(x,y)dA\right]$ $(a = 2m, b = 1\ m)$.

$\varphi(x,y)$	$\ln\varphi(x,y)$	$\frac{1}{A}\iint_D \ln\varphi(x,y)\,d\,A$	$\ln\left[\frac{1}{A}\iint_D \varphi(x,y)dA\right]$	$\frac{1}{A}\iint_D \ln\varphi(x,y)dA \le \ln\left[\frac{1}{A}\iint_D \varphi(x,y)dA\right]$
c (any positive constant)	$\ln c$	$\ln c$	$\ln c$	YES
e^x	x	1	$\ln\left[\frac{1}{2}\left(e^2-1\right)\right] = 1.16$	YES
e^y	y	0.5	$\ln(e-1) = 0.54$	YES
e^{x-y}	$x-y$	0.5	$\ln\left[\frac{1}{2}\left(e^2+e^{-1}-e-1\right)\right] = 0.70$	YES

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the second-law approach, to obtain them, thus building their relation to the second law of thermodynamics. The second-law approach works also for developing new mathematical inequalities.

4. Concluding remarks

As the time tends to infinity,the entropy generation during the heat conduction in adiabatic rectangles evolves into a finite value when the heat conduction comes from the initial temperature distribution, but infinity when the conduction contains any heat source of positive average over the rectangle. Therefore, it is critical to use as less technologies as possible that could yield positive average heat generation and to develop their replacing technologies, for preventing unbounded entropy generation and thus exergy or energy/quality destruction.

The nine inequalities are obtained by applying the second law of thermodynamics to the heat conduction in rectangles: eight being new and the other one being classical. They are capable for examining correctness and accuracy of various studies of heat conduction processes. An application of one of these nine inequalities to the total multiplicity of a macrostate in the Boltzmann entropy concludes that the entropy average over any domain can never be larger than the entropy calculated with the average multiplicity over the domain.

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Author Contributions

X. T and L. W. conceived the project. X. T. and L. W. designed the project. X. T. performed mathematical derivation. X. T. and L. W. analyzed the results. X. T. and L. W. wrote the manuscript. L. W. supervised the study.

Additional Information

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